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NUMERICAL SOLUTION OF FRACTIONAL PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS VIA TWO-DIMENSIONAL FRACTIONAL MUNTZ-LEGENDRE POLYNOMIALS

[1] mahmood shareef, [2] Khalid farhan fazea
[1],[2] Shatrah Universit, College of Engineering , Shatrah Tho-Qar / Iraq.
[1] mahmoodshareef@shu.edu.iq, [2] khalidfarhanfazea@shu.edu.iq

Abstract

We present a numerical method for solving a class of linear and nonlinear fractional partial integro-differential equations (FPIDEs). The purpose of the proposed method is to provide an orthogonal basis for two-dimensional fractional Montz-Legendre polynomials. Based on these polynomials, we approximate unknown functions. We also provide an operational matrix for the fractional derivative in the sense of Caputo to compute fractional derivatives. The proposed approximation, together with the Tao method, transforms the solution of FPIDE equations into the solution of a system of algebraic equations. Finally, to demonstrate the validity and accuracy of the present method, we give some numerical examples.

Keywords: Numerical solution, fractional differential equations, polynomials, Muntz Legend.

الحل العددي للمعادلات التفاضلية التكاملية الجزئية الكسرية من خلال متعددات الحدود الكسرية ثنائية الأبعاد من نوع مونتز ليجندر

محمود شريف[1] ، خالد فرحان فازع[2]
[1],[2] جامعة الشطرة، كلية الهندسة ، العراق / ذي قار
[1] mahmoodshareef@shu.edu.iq 07822923680, [2] khalidfarhanfazea@shu.edu.iq 07802831202

الملخص

نقدم طريقة عددية لحل فئة من المعادلات التفاضلية الجزئية الكسرية الخطية وغير الخطية .(FPIDEs) الهدف الرئيسي من الطريقة المقترحة هو تقديم أساس متعامد لمتعددات حدود Muntz-Legendre الكسرية ثنائية الأبعاد. باستخدام هذه الحدود، نقوم بتقريب الدوال المجهولة. علاوة على ذلك، يتم توفير مصفوفة تشغيلية للمشتقة الكسرية بمعنى Caputo لحساب المشتقات الكسرية. يقلل التقريب المقترح مع طريقة Tau من حل FPIDEs إلى حل نظام من المعادلات الجبرية. أخيرًا: لإظهار صحة ودقة الطريقة الحالية: نقدم بعض الأمثلة العددية.

الكلمات المفتاحية : الحل العددي، المعادلات التفاضلية الجزئية الكسرية، متعددات الحدود، مونتز لجند.

1. Introduction

We consider a class of fractional partial integro-differential equations (FIPDEs) as follows

(1)

$$D_x^{\alpha}u(x,t)+D_t^{\beta}u(x,t)=g(x,t)+\int_0^1\int_0^1 k(x,t,y,s)u^p(y,s)dyds$$

(2)

$$u(x,0)=h_0(x).\ u(0,t)=f_0(t)\ 0<\alpha.\beta\leq 1.\ x.t\in\Omega=(0,1]\times(0,1].$$

where $p\in\mathbb{N}$ and α,β are the Caputo sense's order of the fractional derivatives and the function $g(x,t)\in L^2(\Omega)$: and $k(x,t.y:s)\in L^2(\Omega\times\Omega)$ are known, and $u(x,t)$ is unknown to be determined. Researchers have recently presented a number of numerical methods for solving certain types of fractional integro-differential equations. For example. Yousefi, Javadi, and Babolian [1] used a computational approach for solving fractional integral equations based on Legendre collocation method. Mahdi [2] used the least sqares method aid of Hermite polynomials for solving a linear system of fractional integro-differential equations. Yang. Chen, and Huang [3] used the spectral-collocation method for fractional fredholm integro-differential equations. Saadatmandi and Dehghan [4] used a Legendre collocation method for fractional integro-differential equations. Mojahedfar and Marzabad [5] used normalized Legendre polynomials for solving a class of linear partial fractional Fredholm integro-differential equations. The other numerical methods can be found in [6,7,8,8,9].

In this study, we solve equation (1) numerically by applying two-dimensional fractional Muntz-Legendre polynomials (2D-FMLPs).

The structure of this article is as follows: A quick review of the Caputo fractional derivative is given in Section 2, and this section also includes a review of the one-dimensional fractional Muntz-Legendre polynomials. Two dimensional fractional Muntz-Legendre polynomials are presented in Section 3. We present an operational matrix of fractional derivative in the Caputo sense in Section 4. In Section 5, we use the Tau technique based on 2D-FMLPs to estimate the solution of FPIDEs. To demonstrate the effectiveness and precision of the suggested approach, we provide a few FPIDE examples in Section 6. Finally, Section 7 presents a conclusion..

2. Preliminaries

2.1. Caputo fractional derivative

In this section, we introduce definitions and preliminary qualities of the fractional calculus, which are adopted in this paper.

Definition 1. The Caputo fractional derivative of order α is given by

$$D^{\alpha}u(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t \frac{u^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau; & n-1<\alpha<n,\ n\in\mathbb{N},\\ \dfrac{d^n u(t)}{dt^n}, & \alpha=n, t>0\end{cases}$$

It can easily be shown that

$$D^\alpha C = 0.$$

where C is a constant and

(3) $D^\alpha t^v$

$$= \begin{cases} 0, & v \in \mathbb{N}_0, \text{ and } v <\lceil \alpha], \\ \dfrac{\Gamma(v+1)}{\Gamma(v+1-\alpha)} t^{v-\alpha}, & v \in \mathbb{N}_0 \text{ and } v \geq\lceil \alpha] \text{ or } v \notin \mathbb{N}_0 \text{ and } v >\lceil \alpha] \end{cases}$$

where $[\alpha]$ is the integer part of $\alpha > 0$, and $\mathbb{N}_0 = \{0,1,2,\dots\}$. Also, we have

$$D^\alpha\left(\sum_{i=1}^{n} a_i u_i(t)\right) = \sum_{i=1}^{n} a_i D^\alpha u_i(t).$$

where $a_i, i = 1 \dots\dots n$ are constants.

2.2. One-dimensional fractional Muntz-Legendre polynomials

The one-dimensional fractional Muntz-Legendre polynomials $L_i(t;\alpha)$ on [0,1] are given by [10] :

(4) $$L_i(t:\alpha) = \sum_{k=0}^{i} C_{i,k} t^{k\alpha}.\ C_{i,k} = \frac{(-1)^{i-k}}{\alpha^i k!\,(i-k)!} \prod_{v=0}^{i-1} ((k+v)\alpha + 1).$$

Remark 2.1. According to Eq. (4) the analytic form of $L_i(t;\alpha)$ can be written on the form:

(5) $$L_i(t:\alpha) = \sum_{k=0}^{i} b_{k,i} t^{k\alpha}.$$

where

(6) $$b_{k,i} = \frac{(-1)^{i-k}\Gamma\left(\frac{1}{\alpha} + k + i\right)}{k!\,(i-k)!\,\Gamma\left(\frac{1}{\alpha} + k\right)}.$$

Also. we have

(7) $$L_i(t;\alpha) = P_i^{\left(0;\frac{1}{\alpha}-1\right)}(2t^\alpha - 1),\ \alpha > 0.$$

where $P_i^{(\alpha,\beta)}$ are the Jacobi polynomial with parameters $\alpha, \beta > -1$ [see 10,11]. By Eq. (7) and the recurrence relation between the Jacobi polynomials |11], we can obtain the following recurrence formula:

$$L_{i+1}(t;\alpha) = a_i^\alpha L_i(t;\alpha) - b_i^\alpha L_{i-1}(t;\alpha);\ i = 1,2 \dots$$

where

$$a_i^\alpha = \frac{\left(2i+\frac{1}{\alpha}\right)\left|\left(2i+\frac{1}{\alpha}-1\right)\left(2i+\frac{1}{\alpha}+1\right)(2t^\alpha-1)-\left(\frac{1}{\alpha}-1\right)^2\right.}{2(i+1)\left(i+\frac{1}{\alpha}\right)\left(2i+\frac{1}{\alpha}-1\right)}.$$

$$b_i^\alpha = \frac{i\left(i+\frac{1}{\alpha}-1\right)\left(2i+\frac{1}{\alpha}+1\right)}{(i+1)\left(i+\frac{1}{\alpha}\right)\left(2i+\frac{1}{\alpha}-1\right)}. L_0(t:\alpha)=1.\ L_1(t:\alpha)=\left(\frac{1}{\alpha}+1\right)t^\alpha-\frac{1}{\alpha}.$$

Remark 2.2. The fractional Muntz-Legendre polynomials are orthogonal on the interval [0,1] with the orthogonality relation:

(8) $$\int_0^1 L_i(t;\alpha)L_j(t;\alpha)dt=\frac{1}{2i\alpha+1}\delta_{ij}$$

where δ_{ij} is the Kronecker function.

Definition 2. The expansion of any arbitrary function u(t) that is integrable in [0,1] is as follows:

(9) $$u(t)=\sum_{i=0}^{\infty}u_iL_i(t;\alpha)$$

where

$$u_i=(2i\alpha+1)\int_0^1 u(t)L_i(t:\alpha)dt.\ i=0.1\ldots$$

In practice. only the first $(m+1)$-terms of fractional Muntz-Legendre polynomials are considered. The infinite series of Eq. (9) may then be used to write it as

(10) $$u(t)\simeq u_m(t)=\sum_{i=0}^{m}u_iL_i(t;\alpha)=U^T\phi(t;\alpha)$$

where

(11) $$U=|u_0\cdot u_1\ldots u_m|^T,$$

(12) $$\phi(t;\alpha)=|\,L_0(t;\alpha),L_1(t;\alpha)\ldots L_m(t;\alpha)]^T,$$

Remark 2.3. According to Remark 2.2. we have

(13) $$\int_0^1 \phi(t;\alpha)\phi^T(t;\alpha)dt=\begin{bmatrix}\frac{1}{2(0)\alpha+1} & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{2(1)\alpha+1} & 0 & \cdots & 0\\ 0 & 0 & \ddots & \ddots & \vdots\\ \vdots & \vdots & & \ddots & 0\\ 0 & 0 & \cdots & 0 & \frac{1}{2(m)\alpha+1}\end{bmatrix}.$$

2.3. Error bound

the following theorem shows approximation converges fractional MuntzLegendre polynomials to $u(t)$.

Theorem 2.4. Suppcse $D^{k\alpha}u(t) \in C$ | 0,1] for $k = 0,1$ .... m. If $u_m(t)$ in $Eq.$ (10) is the lest approximaticn to $u(t)$ from $M_{m,\alpha} = \text{span}\{L_0(t;\alpha), L_1(t;\alpha), \ldots . L_m(t;\alpha)\}$, then

$$|u(t) - u_m(t)|_\omega \leq \frac{M_\alpha}{\Gamma(m\alpha+1)\sqrt{2m\alpha+1}}.$$

where $M_\alpha \geq D^{m\alpha}u(t), t \in | \ 0.1]$.

Proof. By applying Taylor's formula (see |12. 13]), then

$$u(t) = \sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} D^{k\alpha}u(0^+) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} D^{m\alpha}u(\xi)\text{: } 0 < \xi < t;\ t \in | \ 0,1$$

Also.

$$u(t) - \sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} D^{k\alpha}u(0^+) \leq \frac{M_\alpha t^{m\alpha}}{\Gamma(m\alpha+1)}$$

On the other hand, $u_m(t) = U^T\phi(t;\alpha)$ is the best approximation to $u(t)$. and $\sum_{k=0}^{m-1} \frac{t^{ko}}{\Gamma(k\alpha+1)} D^{k\alpha}u(0^+) \in M_{m,\alpha}$. So, it can be written as

$$\begin{aligned} |u(t) - u_m(t)|_\omega^2 &\leq \left| u(t) - \sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} D^{k\alpha}u(0^+) \right|_\omega^2 \\ &\leq \frac{M_\alpha^2}{\Gamma(m\alpha+1)^2} \int_0^1 t^{2m\alpha} dt = \frac{M_\alpha^2}{\Gamma(m\alpha+1)^2(2m\alpha+1)}. \end{aligned}$$

Hence.

$$|u(t) - u_m(t)|_\omega \leq \frac{M_\alpha}{\Gamma(m\alpha+1)\sqrt{2m\alpha+1}}.$$

So, the proof is completed.

3. Two-dimensional fractional Muntz-Legendre polynomials

In this section, we define the fractional Muntz-Legendre polynomials in the domain Ω=|0,1]×|0,1] . We can define these polynomials in the following , to reach our goal.

Definition 3. Assume $\{L_n(t;\alpha)\}_{n=0}^{\infty}$ be the one-dimensional fractional Muntz-Legendre polynomials on | 0,1]. We call $\{L_i(x;\alpha)L_j(t;\beta)\}_{i,j=0}^{\infty}$ the two-dimensional fractional Muntz-Legendre polynomials (2D-FMLPs) on $\Omega = [0.1] \times [0.1]$.

Theorem 3.1. Tow-FMLPs ate grthogcnal on Ω.
Proof. For $i \neq p, j \neq q$, we get

$$\int_0^1 \int_0^1 L_i(a:\alpha) L_j(t:\beta) L_p(x:\alpha) L_q(t:\beta) dxdt$$
$$= \left(\int_0^1 L_i(x:\alpha) L_p(x:\alpha) dx \right) \left(\int_0^1 L_j(t:\beta) L_q(t:\beta) dt \right) = 0.$$

and for $i = p. j = q$. we have

$$\int_0^1 \int_0^1 \mid L_i(a;\alpha) \Big]^2 \mid L_j(t;\beta) \Big]^2 dxdt \quad = \left(\int_0^1 [L_i(x,\alpha)]^2 dx \right) \left(\int_0^1 \mid L_j(t;\beta) \Big]^2 dt \right)$$
$$= \frac{1}{2i\alpha + 1} \cdot \frac{1}{2j\beta + 1}.$$

Hence, the proof is completed.
Definition 4. The function $u(x.t)$ that is integrable in $\Omega =\mid 0.1] \times \mid 0.1]$ can be expanded on the form:

(14) $$u(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} L_i(x;\alpha) L_j(t;\beta)$$

where

(15) $$u_{ij} = (2i\alpha + 1)(2j\beta + 1) \int_0^1 \int_0^1 u(x,t) L_i(x;\alpha) L_j(t;\beta) dxdt$$

In practice, only the first $(m_1)(m_2)$-terms of 2D-FMLPs are considered. Then it can be written by the infinite series of Eq.(14) as

(16)

$$u(x,t) \simeq u_{m_1,m_2}(x,t) = \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} u_{ij} L_i(x;\alpha) L_j(t;\beta)$$
$$= U^T(\phi(x;\alpha) \otimes \phi(t;\beta) = U^T \Upsilon(x,t),$$

where ⊗ is the Kronecker product and

(17) $$\phi(x;\alpha) =\mid L_0(x;\alpha), L_1(x;\alpha) \dots L_{m_1-1}(x;\alpha) \Big]^T,$$
$$\phi(t;\beta) = \Big| L_0(t;\beta), L_1(t;\beta) \dots L_{m_2-1}(t;\beta) \Big|^T,$$

$$U=\begin{bmatrix} u_{0,0}\\ u_{0,1}\\ \vdots\\ u_{0,m_2-1}\\ u_{1,0}\\ \vdots\\ u_{i,j-1}\\ u_{i,j}\\ u_{i,j+1}\\ \vdots\\ u_{m_1-1,m_2-2}\\ u_{m_1-1\cdot m_2-1}\end{bmatrix}_{m_1m_2\times 1} : \Upsilon(x,t)=\begin{bmatrix} L_0(x;\alpha)L_0(t;\beta)\\ L_0(x;\alpha)L_1(t;\beta)\\ L_0(x;\alpha)L_{m_2-1}(t;\beta)\\ L_1(x;\alpha)L_0(t;\beta)\\ \vdots\\ L_i(x;\alpha)L_{j-1}(t;\beta)\\ L_i(x;\alpha)L_j(t;\beta)\\ L_i(x;\alpha)L_{j+1}(t;\beta)\\ \vdots\\ L_{m_1-1}(x;\alpha)L_{m_2-2}(t;\beta)\\ L_{m_1-1}(x;\alpha)L_{m_2-1}(t;\beta)\end{bmatrix}_{m_1m_2\times 1}$$

Remark 3.2. A product matrix of FMLPs vectors given by:

$$\Upsilon(x,t)\Upsilon^T(x,t)U \simeq \tilde{U}\Upsilon(x,t),$$

where $\tilde{U}$ is an $(m_1)(m_2)\times(m_1)(m_2)$ product matrix for the vector U as follows:

$$\tilde{U} = | \tilde{U}^{0.0} : \tilde{U}^{0.1} \dots \dot{U}^{0.m_2-1} \cdot \tilde{U}^{1.0} \dots \tilde{U}^{1,m_2-1} \dots \tilde{U}^{m_1-1.0} \dots \tilde{U}^{m_1-1,m_2-1}]^T,$$

where $\tilde{U}^{p,q} : p = 0,1,\dots,m_1-1, q = 0,1,\dots m_2-1$, is an $1\times m_1m_2$ matrix as

$$\hat{U}^{p.q} = [\hat{U}^{p,q}_{0,0} : \hat{U}^{p,q}_{0,1} \dots \hat{U}^{p,q}_{0,m_2-1} \hat{U}^{p.q}_{1,0} \dots \tilde{U}^{p,q}_{1,m_2-1} \dots \hat{U}^{p,q}_{m_1-1,0} \dots \hat{U}^{p.q}_{m_1-1,m_2-1}].$$

and each element of $U^{p.q}$ is obtained as

$$\bar{U}^{p.q}_{k.l} = (2k\alpha+1)(2l\beta+1)\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} u_{ij}g_{ijpqkl.}\, p.k = 0.1\dots.m_1-1,\ \text{q.} = 0,1\dots\dots m_2-1,$$

where g_{ijpqkl} is given by

$$g_{ijpqkl} = \int_0^1\int_0^1 L_i(x;\alpha)L_j(t;\beta)L_p(x;\alpha)L_q(t;\beta)L_k(x;\alpha)L_l(t;\beta)dxdt$$

Remark 3.3. Assume that $f(x,t)\simeq F^T\Upsilon(x,t)$ and $g(x,t)\simeq G^T\Upsilon(x,t)$, where F and G are an $(m_1)(m_2)$-vector as

$$F = [\, f_{0,0}: f_{0,1}, \dots, f_{0m_2-1}, f_{1,0}, \dots, f_{i,j-1}: f_{i,j}, f_{i,j+1}, \cdots, f_{m_1-1,m_2-2}, f_{m_1-1,m_2-1}\,]^T,$$

$$G = [\, g_{0.0}, g_{0,1}, \dots, g_{0,m_2-1}, g_{1,0}, \dots, g_{i,j-1}, g_{i,j}, g_{i,j+1}, \cdots, g_{m_1-1,m_2-2}: g_{m_1-1,m_2-1}\,]^T,$$

where, from Definition 4. each element of F and G is obtained as

$$f_{i,j} = (2i+1)(2j\alpha+1)\int_0^1\int_0^1 f(x,t)L_i(x)L_j(t;\alpha)dxdt.$$

$$g_{i,j} = (2i+1)(2j\alpha+1)\int_0^1\int_0^1 g(x,t)L_i(x)L_j(t;\alpha)dxdt.$$

We can write

$$f(x,t)g(x,t) \simeq A^T\Upsilon(x,t)\Upsilon^T(x,t)B \simeq A^T\tilde{B}\Upsilon(x,t) \simeq B^T\bar{A}\Upsilon(x,t).$$

Lemma 3.4. The nonlinear operator $N(u(x,t)) = u^p(x,t)$, where $u(x,t) \simeq U^T\Upsilon(x,t)$, can te approximated by

$$u^p(x,t) \simeq U^T\tilde{U}^{p-1}\Upsilon(x,t).$$

Proof. From Remark (3.3). we have

$$\begin{aligned} u^2(x,t) &= u(x,t)u(x,t) \simeq U^T\tilde{U}\Upsilon(x,t) \\ u^p(x,t) &= u^2(x,t)u^{p-2}(x,t) \simeq U^T\tilde{U}\Upsilon(x,t)u(x,t)u^{p-3}(x,t) \simeq U^T\tilde{U}\Upsilon(x,t)\Upsilon^T(x,t)Uu^{p-} \\ &\simeq U^T\tilde{U}^2\Upsilon(x,t)u^{p-3}(x,t) \simeq \cdots \simeq U^T\tilde{U}^{p-1}\Upsilon(x,t). \end{aligned}$$

So the proof is completed.

Furthermore, let $k(x.t.y.s)$ be a function of four variables on $\Omega \times \Omega$. It can be approximated with respect to 2D-FMLPs as follows:

$$\begin{aligned} k(x,t,y,s) &\simeq \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1}\sum_{l=0}^{m_4-1} a_{i,j,k,l}L_i(x;\alpha)L_j(t;\beta)L_k(y;\alpha)L_l(s;\beta) \\ &= (\phi^T(x;\alpha)\otimes\phi^T(t;\beta))K(\phi(y:\alpha)\otimes\phi(s;\beta)) \\ &= \Upsilon^T(x,t)K\Upsilon(y,s) \end{aligned}$$

where

$$\phi(x;\alpha) = [\, L_0(x;\alpha), L_1(x;\alpha) \dots L_{m_1-1}(x;\alpha)\,]^T,$$

$$\phi(t;\beta) = [\, L_0(t;\beta), L_1(t;\beta) \dots L_{m_2-1}(t;\beta)\,]^T,$$

$$\phi(y;\alpha) = [\, L_0(y;\alpha), L_1(y;\alpha) \dots L_{m_3-1}(y;\alpha)\,]^T.$$

$$\phi(s:\beta) = [\, L_0(s;\beta), L_1(s;\beta) \dots L_{m_4-1}(s:\beta)\,]^T.$$

and K is an $(m_1m_2 \times m_3m_4)$-matrix as follows:

$$K = \begin{bmatrix} K^{0,0} & K^{0,1} & \cdots & K^{0,m_3-1} \\ K^{1,0} & K^{1,1} & \cdots & K^{1,m_3-1} \\ \vdots & \vdots & \vdots & \vdots \\ K^{m_1-1,0} & K^{m_1-1,1} & \cdots & K^{m_1-1,m_3-1} \end{bmatrix}.$$

$$K = \begin{bmatrix} K^{0,0} & K^{0,1} & \cdots & K^{0,m_3-1} \\ K^{1,0} & K^{1,1} & \cdots & K^{1,m_3-1} \\ \vdots & \vdots & & \vdots \\ K^{m_1-1,0} & K^{m_1-1,1} & \cdots & K^{m_1-1,m_3-1} \end{bmatrix}$$

where each block of $K(K^{p,q} \cdot p = 0.1 \ldots \ldots m_1 - 1. q = 0.1 \ldots \ldots m_3 - 1)$ is an $m_2 \times m_4$-matrix. Here. if we assume that $K^{p,q} = \{k_{i.j}^{p,q}\}, i = 0.1 \ldots, m_2 - 1, j = 0.1 \ldots, m_4 - 1$ then

$$K_{i,j}^{p,q} = (2p\alpha + 1)(2i\beta + 1)(2q\alpha + 1)(2j\beta + 1) \int_0^1 \int_0^1 \int_0^1 \int_0^1 k(x,t,y - s) L_p(x;\alpha) L_i(t;\beta) L_q(y;\alpha) L_j(s;\beta) dxdtdyds.$$

Remark 3.5. By using the orthogonality property given in Theorem 3.1. we have

$$(21) \qquad \int_0^1 \int_0^1 \Upsilon(x,t)\Upsilon^T(x,t)dxdt = \begin{bmatrix} E_0 & 0 & 0 & \cdots & 0 \\ 0 & E_1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & E_{m_1-1} \end{bmatrix}_{m_1 m_2 \times m_1 m_2} = E,$$

where each block of $E(E_i, i = 0,1, \ldots m_1 - 1)$ is an $(m_2 \times m_2)$-matrix as follows:

$$E_i = \begin{bmatrix} \int_0^1\int_0^1 L_i(x;\alpha)L_0(t;\beta)L_i(x;\alpha)L_0(t;\beta)dxdt & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \int_0^1\int_0^1 L_i(x;\alpha)L_{m_2-1}(t;\beta)L_i(x;\alpha)L_{m_2-1}(t;\beta)dxdt \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(2i\alpha+1)(2(0)\beta+1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{(2i\alpha+1)(2(1)\beta+1)} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{(2i\alpha+1)(2(m_2-1)\beta+1)} \end{bmatrix}_{m_2 \times m_2}.$$

In this paper for convergence, we assumed that m_1=m_2=m_3=m_4=m.

The presence of uniqueness and convergence analyses is stated in the following theorems.

Theorem 3.6. If the double series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} L_i(x:\alpha) L_j(t:\beta)$$

converges uniformly to $u(x,t)$ on the $\Omega =| 0,1] \times| 0,1]$, then

$$a_{ij} = (2i\alpha + 1)(2j\beta + 1)\int_0^1\int_0^1 u(x,t)L_i(x:\alpha)L_j(t:\beta)dxdt.$$

Proof. Set

$$u(x.t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}L_i(x:\alpha)L_j(t;\beta).$$

and let m, n are fixed. Then we can write

$$\begin{aligned}\int_0^1\int_0^1 u(x,t)L_m(x;\alpha)L_n(t;\beta)dxdt &= \int_0^1\int_0^1\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}L_i(x;\alpha)L_j(t;\beta)\right)L_m(x;\alpha)L_n(t;\beta)dxdt \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\left(\int_0^1 L_i(x;\alpha)L_m(x;\alpha)dx\right)\left(\int_0^1 L_j(t;\beta)L_n(t;\beta)dt\right) \\ &= a_{ij}\left(\int_0^1 L_i^2(x;\alpha)dx\right)\left(\int_0^1 L_j^2(t;\beta)dt\right) \\ &= a_{ij}\left(\frac{1}{2i\alpha+1}\right)\left(\frac{1}{2j\beta+1}\right).\end{aligned}$$

Therefore, the proof is completed.

Lemma 3.7. If $u(x.t)$ is a continuous functicn on $| 0.1] \times [0.1]$ and the doutle seties

(22) $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}L_i(x;\alpha)L_j(t:\beta)$$

converges uniformly to $u(x,t)$, then Eq. (22) is the FMLPs expansion of $u(x,t)$.

Procf. (By contradiction) Let

$$u(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij}L_i(x;\alpha)L_j(t;\beta)$$

and

$$u(x,t) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}L_i(x;\alpha)L_j(t;\beta).$$

Then there is at least one coefficient such that $a_{nm} \neq b_{nm}$. However.

$$b_{nm} = (2n\alpha + 1)(2m\beta + 1)\int_0^1\int_0^1 u(x,t)L_n(x;\alpha)L_m(t;\beta)dxdt = a_{nm}$$

Lemma 3.8. If tuc continuous functions defined on $[0.1] \times [0.1]$ have the identical FMLPs expansions, then these two functions are identical.

Proof. Let

$$u(x,t) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}L_i(x:\alpha)L_j(t;\beta).$$

and

$$v(x,t) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}L_i(x;\alpha)L_j(t;\beta)$$

Then. we have

$$\begin{aligned} u(x,t) - v(x,t) &\sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left(a_{ij} - a_{ij}\right)L_i(x;\alpha)L_j(t;\beta) = 0 \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} 0L_i(x;\alpha)L_j(t:\beta). \end{aligned}$$

Thus

$$u(x,t) - v(x,t) = 0.$$

So. the proof is completed.

Theorem 3.9. If the 2DFMLPs expansion of a function $u(x,t)$ converges uniformly, then every 2D-FMLPs exponsion converges to $u(x,t)$.
Proof. It is the result of the Lemma (3.7) and Lemma (3.8).

4. FMLPs operational matrix of fractional derivatives

In this section, we introduce an operational matrix of fractional derivative. First, we consider the following lemma.
Lemma 4.1. Let $L_i\ (t;\alpha)$ be a FMLP, then the Caputo fractional derivative of $L_i\ (t:\alpha)$ of order γ>0 can be obtained as

$$D^{\gamma}L_i(t;\alpha) = \sum_{k=0}^{i} b'_{k,i}\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\gamma+1)}t^{k\alpha-\gamma}:$$

where

$$b'_{k,i} = \begin{cases} 0. & k\alpha \in \mathbb{N}_0 \text{ and } k\alpha < \gamma. \\ b_{k,i}. & k\alpha \in \mathbb{N}_0 \text{ and } k\alpha \geq \gamma \text{ or } k\alpha \notin \mathbb{N}_0 \text{ and } k\alpha \geq\mid \gamma]. \end{cases}$$

and $b_{k,i}$ is found in Eq. (6).
Proof. From relations (3) and (5), the proof of the lemma is clear.

Now, the operational matrix of fractional derivative of FMLPs gives by the below theorem .

Theorem 4.2. Let $\phi(x:\alpha)$ is the fracticnal Muntz-Legendre vector as

$$\phi(x;\alpha) =| L_0(x:\alpha), L_1(x:\alpha), \ldots, L_m(x;\alpha)]^T.$$

then

$$D^\gamma \phi(x:\alpha) = D^{(\gamma)} \phi(x;\alpha)$$

where $D^{(\gamma)}$ is the $(m+1) \times (m+1)$ cperational matrix of fracticnal derivative of order $\gamma > 0$. Then an elements of $D^{(\gamma)}$ are obtained as

$$D_{ij}^{(\gamma)} = (2j\alpha+1) \sum_{k=0}^{i} \sum_{s=0}^{j} b_{s,j} b'_{k,i} \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\gamma+1)} \cdot \frac{1}{(k+s)\alpha-\gamma+1}.$$

where

$$b'_{k,i} = \begin{cases} 0. & k\alpha \in \mathbb{N}_0 \text{ and } k\alpha < \gamma. \\ b_{k,i}. & k\alpha \in \mathbb{N}_0 \text{ and } k\alpha \geq \gamma \text{ or } k\alpha \notin \mathbb{N}_0 \text{ and } k\alpha \geq| \gamma]. \end{cases}$$

Proof: According to Lemma 4.1, we have

$$D^\gamma L_i(t:\alpha) = \sum_{k=0}^{i} b'_{k\cdot i} \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+1-\gamma)} t^{k\alpha-\gamma}. \tag{23}$$

On the other hand, approximative $t^{k\alpha-\gamma}$ by using Eq. (10) gives the following relation:

$$t^{k\alpha-\gamma} \simeq \sum_{j=0}^{m} a_{k,j} \ L_j(t;\alpha) \tag{24}$$

$$\begin{aligned} a_{k,j} &= (2j\alpha+1) \int_0^1 t^{k\alpha-\gamma} L_j(x;\alpha) dt \\ &= (2j\alpha+1) \sum_{s=0}^{j} b_{s,j} \int_0^1 t^{k\alpha-\gamma+s\alpha} dt \\ &= (2j\alpha+1) \sum_{s=0}^{j} b_{s,j} \frac{1}{k\alpha-\gamma+s\alpha+1}. \end{aligned}$$

Now. according to Eq. (23)-(25). we can write

$$D^\gamma(t;\alpha) = (2j\alpha+1) \sum_{j=0}^{m} \sum_{k=0}^{i} \sum_{s=0}^{j} b_{s,j} b'_{k,i} \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+1-\gamma)} \cdot \frac{1}{(k\alpha-\gamma+s\alpha+1)} L_j(t;\alpha).$$

Thus.

$$D_{ij}^{\gamma} = (2j\alpha+1) \sum_{k=0}^{i} \sum_{s=0}^{j} b_{s,j} b'_{k,i} \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\gamma+1)} \cdot \frac{1}{(k+s)\alpha-\gamma+1},$$

5. Approximate solution to the FPIDEs

In this section, we used the operational matrices of fractional derivative and product of FMLPs, to approximate the solution of the nonlinear and linear fractional partial integro-differential equation (1) with condition (2) . Thus, using Eqs. (16), (18), and Theorem 4.2, we may estimate the known functions, the unknown function u(x,t), and the differentials D_x^α u(x,t),D_t^β u(x,t) in the following ways:

(26) $$u(x,t) \simeq U^T \Upsilon(x,t):$$

(27) $$g(x,t) \simeq G^T \Upsilon(x,t);$$

(28) $$k(x,t,y,s) \simeq \Upsilon^T(x,t) K \Upsilon(y,s)$$

where G is an (m^2)-vector as follows:

$$G = \left| g_{0,0} \cdot g_{0,1} \cdots\cdots, g_{0,m-1} \cdot g_{1,0} \cdots\cdots, g_{i,j-1} \cdot g_{i,j} \cdot g_{i,j+1}, \cdots, g_{m-1,m-2} : g_{m-1,m-1} \right|^T.$$

and the elements of G are similar to the elements of U in Eq. (15) as follows:

$$g_{i,j} = (2i\alpha + 1)(2j\beta + 1) \int_0^1 \int_0^1 g(x,t) L_i(x;\alpha) L_j(t;\beta) dxdt.$$

and K is an $(m^2 \times m^2)$-matrix. which is defined in Eq. (19). Furthermore, the unknown function $u^p(x,t)$ we approximate as

(29) $$u^p(x,t) \simeq U^T \tilde{U}^{p-1} \Upsilon(x,t).$$

Also, the Caputo fractional derivatives of order $\alpha, \beta > 0$ of $u(x,t)$ is given by

(30) $$D_x^\alpha u(x,t) \simeq U^T D_x^\alpha \Upsilon(x,t) \simeq U^T \mathrm{D}_x^\alpha \Upsilon(x,t); \quad \mathrm{D}_x^\alpha = D_x^{(\alpha)} \otimes I_m;$$

(31) $$D_t^\beta u(x,t) \simeq U^T D_t^\beta \Upsilon(x,t) \simeq U^T \mathrm{D}_t^\beta \Upsilon(x,t); \quad \mathrm{D}_t^\beta = I_m \otimes D_t^{(\beta)}:$$

where $D_x^{(\alpha)}$ and $D_t^{(\beta)}$ are the Caputo fractional derivative matrices of $\phi(x,t)$, which are obtained in Theorem 4.2 and I_m is the identity matrix of order m. Substituting Eqs. (26)-(31) in Eq. (1): we have

$$\begin{aligned} U^T(\mathrm{D}_x^\alpha + \mathrm{D}_t^\beta)\Upsilon(x,t) &= G^T\Upsilon(x,t) + \int_0^1 \int_0^1 U^T \bar{U}^{p-1} \Upsilon(y,s) \Upsilon^T(y.s) K^T \Upsilon(x,t) dyds \\ &= G^T\Upsilon(x,t) + U^T \tilde{U}^{p-1} \left(\int_0^1 \int_0^1 \Upsilon(y,s)\Upsilon^T(y,s) dyds \right) K^T \Upsilon(x,t) \end{aligned}$$

(32) $$= G^T\Upsilon(x,t) + U^T \tilde{U}^{p-1} E K^T \Upsilon(x,t).$$

where E is an $(m^2) \times (m^2)$-matrix with the elements given in the Remark 3.5. Now, by a Tau method in |14], we should generate $m^2 - 2m + 1$ from Eq.(32) by making inner product used $L_i(x;\alpha) L_j(t;\beta)$ (for $i,j = 0,1 \ldots, m-2$) as follows:

(33) $$U^T \left(\mathrm{D}_x^\alpha + \mathrm{D}_t^\beta \right) B = \left(G^T + U^T \tilde{U}^{p-1} E K^T \right) B$$

where

$$B=\int_0^1\int_0^1 L_i(x;\alpha)L_j(t;\beta)\Upsilon(x,t)dxdt.$$

On the other hand, using the orthogoality property given in Theorem 3.1. we can write:

$$B=\int_0^1\int_0^1 L_i(x;\alpha)L_j(t;\beta)\Upsilon(x,t)dxdt=\begin{bmatrix}0\\0\\\vdots\\0\\0\\\vdots\\\frac{1}{(2i\alpha+1)(2j\beta+1)}\\\vdots\\0\\0\end{bmatrix},i$$
$$=0,1\ldots m-2$$

Therefore. we rewrite Eq. (33) as follows:

(34) $$U^T\left(\mathrm{D}_x^\alpha+\mathrm{D}_t^\beta-\tilde{U}^{p-1}EK^T\right)=G^T.$$

Also. for the initial conditions (2), we have

(35) $$U^T(I_m\otimes\phi(0;\beta))=H_0^T.\ h_0(x)=H_0^T\phi(x;\alpha),\ i=0,1\ldots.2.$$

(36) $$U^T(\phi(0;\alpha)\otimes I_m)=F_0^T:\ f_0(t)=F_0^T\phi(t;\beta),\ j=0.1\ldots\ldots m-1.$$

Now, we have a algebraic equations m^2-2m+1 of the Eq. (34) together with the $2m-1$ alebraic equations of the Eqs. (35) and (36). After solving this matrix system by using Mathematica software, we can find the m^2 elements of the unknown matrix U and approximate solution

$$u_m(x,t)\simeq U^T\Upsilon(x,t).$$

6. Numerical illustration

In this section, we introduce some examples of nonlinear and linear FPIDEs to show the efficiency of the proposed method. The results will be compared with the exact solutions. The accuracy of the present method is estimated by the absolute error E_m and maximum absolute error ϵ_m are given by:

$$\begin{aligned} E_m &= u(\theta_i, \beta_j) - u_{m \cdot n}(\theta_i, \beta_j), \\ \epsilon_m &= \max\{u(\theta_i, \beta_j) - u_m(\theta_i, \beta_j)\}. \end{aligned}$$

where θ_i and β_j are defined as

$$(\theta_i, \beta_j) = \left(x_i^{\frac{1}{\alpha}}, t_j^{\frac{1}{\beta}} \right), \ i = 0,1, \dots, m, \ j = 0,1, \dots, n,$$

where x_i and t_j are Chebyshev-Gauss-Lobatto points with the following relations:

$$x_i = \frac{1}{2} - \frac{1}{2}\cos\frac{\pi i}{m} \ t_j = \frac{1}{2} - \frac{1}{2}\cos\frac{\pi j}{n}.$$

Example 1.

Consider the linear FPIDE

$$D_x^{\alpha} u(x,t) + D_t^{\beta} u(x,t) = g(x,t) + \int_0^1 \int_0^1 xtysu(y,s)dyds.$$

$$u(x,0) = u(0,t) = 0. \ 0 < \alpha, \beta \leq 1. \ (x,t) \in (0,1] \times (0,1].$$

where

$$g(x,t) = \frac{1}{\Gamma(2-\alpha)} t x^{1-\alpha} + \frac{1}{\Gamma(2-\beta)} x t^{1-\beta} - \frac{xt}{9}.$$

The exact solution is $u(x,t) = xt$. By using the present method to solve this problem with $m = 3$. for $\alpha = \beta = \frac{1}{2}$, we get $\epsilon_3 = 2.0646e - 09$. and

$$U = \begin{bmatrix} 2.5000e-01 \\ 2.0000e-01 \\ 5.0000e-02 \\ 2.0000e-01 \\ 4.0000e-02 \\ 5.0000e-02 \\ 4.0000e-02 \\ 1.0000e-02 \end{bmatrix}$$

Then, the approximation solution is given as

$$u_3(x,t) = U^T\gamma(x,t) = \begin{bmatrix} 1 \\ 3t^{\frac{1}{2}} - 2 \\ 10t - 12t^{\frac{1}{2}} + 3 \\ 3x^{\frac{1}{2}} - 2 \\ \left(3t^{\frac{1}{2}} - 2\right)\left(3x^{\frac{1}{2}} - 2\right) \\ 10tx - 12tx^{\frac{1}{2}} + 3 \\ \left(3t^{\frac{1}{2}} - 2\right)\left(10x - 12x^{\frac{1}{2}} + 3\right) \\ \left(10t - 12t^{\frac{1}{2}} + 3\right)\left(10x - 12x^{\frac{1}{2}} + 3\right) \end{bmatrix} = x(t).$$

which is actually the exact solution. To get less error. we put $m = 3$. In this case. we get $\epsilon_4 = 1.6653e - 16$, and

$$U = \begin{bmatrix} 2.5000e-01 & & 2.5000e-01 \\ 2000e-01 & & 2000e-01 \\ 5000e-02 & & 5000e-02 \\ -1.6523e-16 & & 0 \\ 2.000001e & \simeq & 2.000001e \\ 1.6000e-01 & & 1.6000e-01 \\ 4.0000e-02 & & 4.0000e-02 \\ -2.9560e-16 & & 0 \\ 5.0000e-02 & & 5.0000e-02 \\ 4.0000e-02 & & 4.0000e-02 \\ 1.0000e-02 & & 1.0000e-02 \\ -8.9755e-17 & & 0 \\ -1.8308e-17 & & 0 \\ -1.6962e-16 & & 0 \\ -4.8497e-17 & & 0 \\ 3.9177e-17 & & 0 \end{bmatrix}$$

So, the approximation solution is obtained as

$$u(x.t) = U^T\gamma(z.t) = xt.$$

Example 2.

Consider the linear and nonlinear FPIDE

$$D_x^{\alpha}u(x,t) + D_x^{\beta}u(x,t) = g(t,x) + \int_0^1\int_0^1 (x^2 - t^2)ysu^p\,(y;s)dyds$$

$$u(x,0) = u(0,t) = 0; \quad 0 < \alpha, \beta < 1; \quad (x,t) \in (0,1]$$

where

$$g(x,t) = \begin{cases} \frac{-2}{\Gamma(3-\alpha)} t x^{2-\alpha} + (1-x^2)\frac{t^{1-\beta}}{\Gamma(2-\beta)} - \frac{(x^2-t^2}{12}, & \text{for} \quad p=1, \quad \text{(linear form)}, \\ \frac{-2}{\Gamma(3-\alpha)} t x^{2-\alpha} + (1-x^2)\frac{t^{1-\beta}}{\Gamma(2-\beta)} - \frac{(x^2-t^2}{12}, & \text{for} \quad p=2, \quad \text{(nonlinear form)}. \end{cases}$$

The exact solution is $u(x,t) = (1-x^2)t$, the results are shown in Tables 1. 2 and Figures 1. 2. 3. Table1 shows the maximum absolute error ε_m for various values of m , and $\alpha = \beta = \frac{2}{10}; \frac{5}{10}; \frac{7}{10}; \frac{8}{10}$ for $p = 1$. Table 2 shows the ϵ_m for various values of m. and $\alpha = \beta = \frac{1}{2}; \frac{1}{3}; \frac{2}{3}$ for $p = 2$. Figure 1 shows the comparison between the exact solution and the approximate solution with the absolute error E_{10} for $\alpha = \beta = \frac{7}{10}; \frac{8}{10};$ for $p = 1$. Figure 2 shows the comparison between the exact solution and the approximate solution with the absolute error E_6 for $\alpha = \beta = \frac{2}{3}$ for $p = 2$. In addition, Figure 3 shows the logarithmic graphs of ϵ_m (log10m) for $\alpha = \beta = \frac{2}{10}; \frac{5}{10}; \frac{7}{10}; \frac{8}{10}$ and $p = 1$ for p = 1.

7 . Conclusion

In this paper, we applied a basis of 2-dimensional fractional Muntz-Legendre polynomials to get the numerical solution of nonlinear and linear FPIDES. To obtained the unknown coefficients 2D-FMLPs, we used the operational matrices of product and fractional derivatives of FMLPs together and the Tau method. The efficiency and accuracy of the proposed method showed the results of the numerical examples and the comparison with exact solution .

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ε_m	$\alpha=\beta=\frac{2}{10}$	$\alpha=\beta=\frac{3}{10}$	$\alpha=\beta=\frac{7}{10}$	$\alpha=\beta=\frac{8}{10}$
ε_2	$1.4065e-01$	$3.4883e-01$	$4.4329e-01$	$4.7447e-01$
ε_3	$8.4210e-02$	$6.3605e-02$	$5.0996e-02$	$3.9421e-02$
ε_4	$3.4198e-02$	$8.0550e-03$	$5.6700e-03$	$7.7979e-03$
ε_5	$1.2817e-02$	$1.2212e-15$	$3.2066e-03$	$4.9242e-03$
ε_6	$3.2974e-03$	$1.1102e-15$	$1.8086e-03$	$3.0440e-03$
ε_7	$7.6160e-04$	$8.6271e-10$	$1.1060e-03$	$1.9823e-03$
ε_8	$1.2750e-04$	$6.6613e-16$	$7.5686e-04$	$1.3926e-03$
ε_9	$1.3781e-05$	$1.9984e-15$	$5.3101e-04$	$1.0315e-03$
ε_{10}	$2.2264e-07$	$2.3629e-15$	$3.8419e-04$	$7.8188e-04$

TABLE 2. Maximum absolute error for $p=2$ of Example 2.

ε_m	$\alpha=\beta=\frac{1}{2}$	$\alpha=\beta=\frac{1}{3}$	$\alpha=\beta=\frac{2}{3}$
ε_2	$3.4883e-01$	$2.4429e-01$	$4.3037e-01$
ε_3	$1.0004e-01$	$1.0413e-01$	$6.1318e-02$
ε_4	$1.6960e-02$	$3.7221e-02$	$7.8730e-03$
ε_5	$7.4165e-03$	$2.2290e-02$	$6.9215e-03$
ε_6	$7.5098e-03$	$8.4156e-03$	$6.5671e-03$

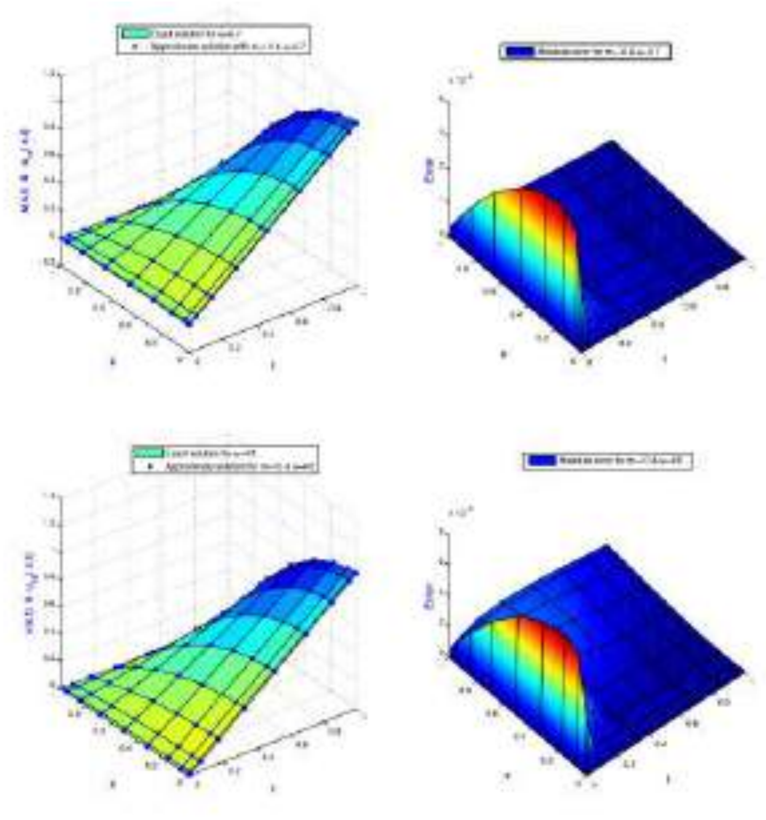

FIGURE 1. Comparison between the exact solution and the approximate solution with absolute error function E_{10} for $\alpha=\beta=\frac{7}{10},\frac{8}{10}$ for $p=1$ of Example 2.

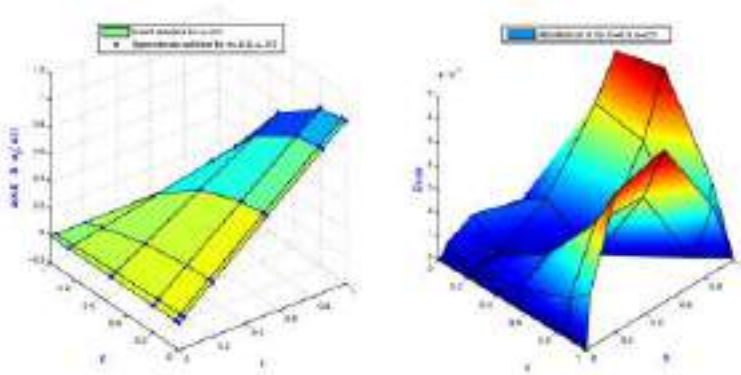

FIGURE 2. Comparison between the exact solution and the approximate solution with absolute error function E_4 for $\alpha = \beta = \frac{2}{3}$ for $p = 2$ of Example 2.

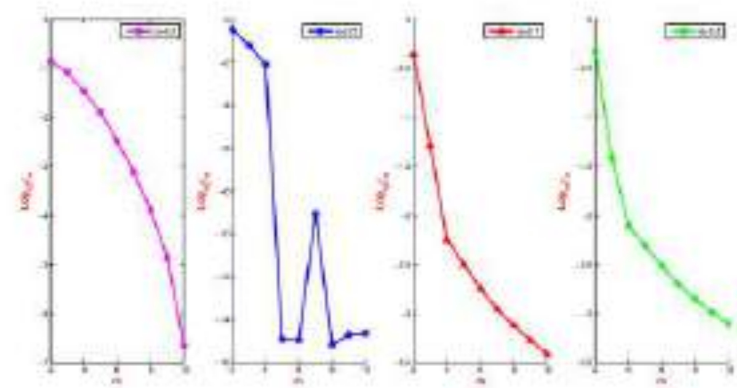

FIGURE 3. The maximum absolute error convergence for $p = 1$ and for various values of α of Example2.